AD-A070 132 SOUTH CAROLINA UNIV COLUMBIA DEPT OF MATHEMATICS COM--ETC F/G 20/3
ASYMPTOTIC BOUNDS FOR SOLUTIONS TO A SYSTEM OF DAMPED INTEGRODI--ETC(U)
MAY 79 F BLOOM AFOSR-77-3396
UNCLASSIFIED AFOSR-TR-79-0723 NL

1 OF 1
AD
A070132

END
DATE
FILMED
8-79
DDC

AFOSR-TR- 79-0723 to appear in the J. Math. Anal. ~~Applic~~

Asymptotic Bounds for Solutions to a System of Damped Integrodifferential Equations of Electromagnetic Theory*

by

Frederick Bloom

Department of Mathematics, Computer Science, and Statistics
University of South Carolina
Columbia, S.C. 29208.

410442

See 1473 in back.

*Research supported in part by AFOSR Grant 77-3396

1. Introduction

As has been recently demonstrated [1] a system of integrodifferential equations governs the evolution of the components of the electric displacement field in a simple class of rigid holohedral isotropic dielectrics of the type introduced by Toupin and Rivlin in [2]. More specifically, we consider the following situation: Let $\Omega \subseteq R^3$ be a bounded region filled with a nonconducting material dielectric substance and assume that $\partial\Omega$, the boundary of Ω, is smooth enough to admit of applications of the divergence theorem. Denote by $\underline{E}, \underline{B}, \underline{P}$ and $\underline{D}$, respectively, the electric field vector, the magnetic flux density, the polarization vector, and the electric displacement vector in Ω; the fields $\underline{E}$ and $\underline{D}$ are related by $\underline{D} = \epsilon_o \underline{E} + \underline{P}$, $\epsilon_o > 0$ a physical constant. By defining, in the usual manner, the magnetic intensity $\underline{H} = \mu_o^{-1} \underline{B}$, where $\mu_o > 0$ satisfies $\epsilon_o \mu_o = c^{-2}$ ($c \equiv$ speed of light in a vacuum) the differential forms of Maxwell's equations in a Lorentz reference frame (x^i, t) become

$$(1.1) \qquad \begin{cases} \dfrac{\partial B}{\partial t} + \text{curl } \underline{E} = \underline{0}, \ \text{div } \underline{B} = 0 \\ \text{curl } \underline{H} - \dfrac{\partial D}{\partial t} = \underline{0}, \ \text{div } \underline{D} = 0 \end{cases}$$

provided that the densities of free current and free charge vanish in Ω, the magnetization is zero in Ω, and the medium is nondeformable (rigid dielectric). To obtain a determinate set of equations for the fields which appear in Maxwell's equations a set of constitutive relations among these fields must be specified and in the theory of rigid nonconducting material dielectrics there exists a hierachy of such constitutive assumptions of increasing complexity. The simplest constitutive assumption possible corresponds to the situation where the dielectric is a vacuum so that $\underline{\Gamma} = \underline{C}$ and $\underline{D} = \epsilon_o E$, $\underline{H} = \mu_o^{-1}\underline{B}$. In [3] and [4] this author has treated the evolution equations associated with the Maxwell-Hopkinson Dielectric in which the constitutive relation between $\underline{D}$ and $\underline{E}$ assumes the form

$$D(\underline{x},t) = \epsilon\underline{E}(\underline{x},t) + \int_{-\infty}^{t} \phi(t-\tau)\underline{E}(\underline{x},\tau)d\tau\,,\ \epsilon > 0$$
$$(\underline{x},t) \in \Omega \times (-\infty,T),\ T > 0,$$

with $|\phi|$ a monotonically decreasing function. The Maxwell-Hopkinson theory retains the simple relation $\underline{H} = \mu_o^{-1}\ \underline{B}$ between the magnetic intensity and magnetic flux density and thus does not take into account the possible influences of magnetic memory effects. Constitutive relations generalizing those of Maxwell-Hopkinson in several directions, and allowing for an understanding of phenomena such as the Faraday effect in dielectrics, were put forth in 1960 by Toupin and Rivlin (op.cit.). One such set of constitutive equations, for a dielectric with holohedral symmetry (i.e., a dielectric which admits the full orthogonal group as its group of material symmetry transformations) has the form

$$(1.2.)\quad \begin{cases} \underline{D}(\underline{x},t) = \sum\limits_{j=o}^{n} a_j\underline{E}^{(j)}(\underline{x},t) + \int_{-\infty}^{t} \phi(t-\tau)\underline{E}(\underline{x},\tau)d\tau \\ \underline{H}(\underline{x},t) = \sum\limits_{j=o}^{n} b_j\underline{B}^{(j)}(\underline{x},t) + \int_{-\infty}^{t} \psi(t-\tau)\underline{B}(\underline{x},\tau)d\tau \end{cases}$$

where the superscripts denote differentiation with respect to the time parameter and the coefficients a_j, b_j are constants; whereas equations (3.2) still effect an priori separation of electric and magnetic effects they now allow for consideration of dielectric materials exhibiting magnetic memory and may be viewed as a linearized version of a more general theory introduced by Volterra in 1912 [5] to treat the case where the dielectric substance is anisotropic, nonlinear, and magnetized, viz:

$$(1.3)\qquad \begin{cases} \underline{D}(\underline{x},t) = \underline{\epsilon}\cdot\underline{E}(\underline{x},t) + \underset{-\infty}{\overset{t}{\underline{\mathcal{D}}}}(\underline{E}(\underline{x},\tau)) \\ \underline{B}(\underline{x},t) = \underline{\mu}\cdot\underline{H}(\underline{x},t) + \underset{-\infty}{\overset{t}{\underline{\mathcal{B}}}}(\underline{H}(\underline{x},\tau)) \end{cases}$$

where ϵ,μ are constant second-order tensors; the constitutive relations (1.2) follow from the set delineated in (1.3) when, among other assumptions, it is assumed that the functionals $\underline{\mathcal{D}},\underline{\mathcal{B}}$ are linear and isotropic.

In [1] we have studied various consequences of the constitutive hypothesis (1.2) under the simplifying assumptions that $a_j = b_j = 0$, $j \ge 1$ and that the past histories of the electric and magnetic fields are of the form

$$(1.4)\qquad \begin{aligned} \underline{E}(\underline{x},t) &= \begin{cases} \underline{0}, & -\infty < t \le -t_h \\ \underline{E}_h(\underline{x},t), & -t_h < t < 0 \end{cases} \\ \underline{B}(\underline{x},t) &= \begin{cases} \underline{0}, & -\infty < t \le -t_h \\ \underline{B}_h(\underline{x},t), & -t_h < t < 0 \end{cases} \end{aligned}$$

for some $t_h > 0$. In particular for memory functions ϕ,ψ which are sufficiently smooth on $(-t_h,\infty)$ we have the following

<u>Lemma</u> [1]: The evolution of the electric displacement field $\underline{D}(\underline{x},t)$ in any holohedral isotropic dielectric (which conforms to the constitutive hypothesis (1.2) with $a_j = b_j = 0$, $j \ge 1$ and past histories of the form (1.4), for some $t_h > 0$, is governed by a system of damped integrodifferential equations of the form

$$\frac{\partial^2 D_i}{\partial t^2} + \Psi(0)\frac{\partial D_i}{\partial t} + \dot{\Psi}(0)[D_i - c_o \nabla^2 D_i]$$

$$+ \int_{-t_h}^{t} [\dot{\Psi}(t-\tau) D_i(\tau) - \left(\frac{b_o}{a_o}\right)\Phi(t-\tau)\nabla^2 D_i(\tau)] d\tau \tag{1.5}$$

$$= 0, \text{ in } \Omega,\ i = 1,2,3,\ c_o = b_o / a_o \dot{\Psi}(0)$$

provided $\underline{D}_h^+ (\underline{x}, -t_h) = \underline{0}$ in Ω and $\dot{\Psi}(0) \neq 0$. In (1.5) $\Phi(t)$ is given in terms of the memory function $\phi(t)$ via the recursion relations

$$\begin{cases} \Phi(t) = \sum\limits_{n=1}^{\infty} (-1)^n \phi^n(t), \ t \geq 0 \\ \phi^1(t) = a_o^{-1} \phi(t), \ \phi^n(t) = \int_{-t_h}^{t} \phi^1(t-\tau)\phi^{n-1}(\tau) d\tau \end{cases}$$

for $n \geq 2$, with a similar definition for $\Psi(t)$ in terms of $\psi(0)$. We assume that $a_o > 0$, $b_o > 0$; it can be shown that $\Psi(0) = -b_o^{-1}\psi(0)$ and thus we assume $\psi(0) < 0$ so that the coefficient of $\frac{\partial D_i}{\partial t}$ in (1.5), i.e., $\Psi(0) > 0$.

<u>Remark</u> The system of integrodifferential equations (1.5), for the components of the electric displacement field, is obtained by combining the constitutive relations (1.2) (with $a_j = b_j = 0$, $j \geq 1$ and past histories of the form (1.4)) with the inverted constitutive equations, giving $\underline{E}$ and $\underline{B}$ in terms of $\underline{D}$ and $\underline{H}$, respectively, Maxwell's equations (1.1), and the vector identity

$$\Delta \underline{V}(\underline{x}) = \text{grad } (\text{div } \underline{V}(\underline{x})) - \text{curl curl } \underline{V}(\underline{x})$$

which is valid $\forall \underline{x} \in \Omega$ for any vector field $\underline{V}(\cdot)$ which is sufficiently smooth on Ω; the constitutive relations (1.2) are inverted by the usual technique of successive approximations. For the details of the computation we refer to [1, §3].

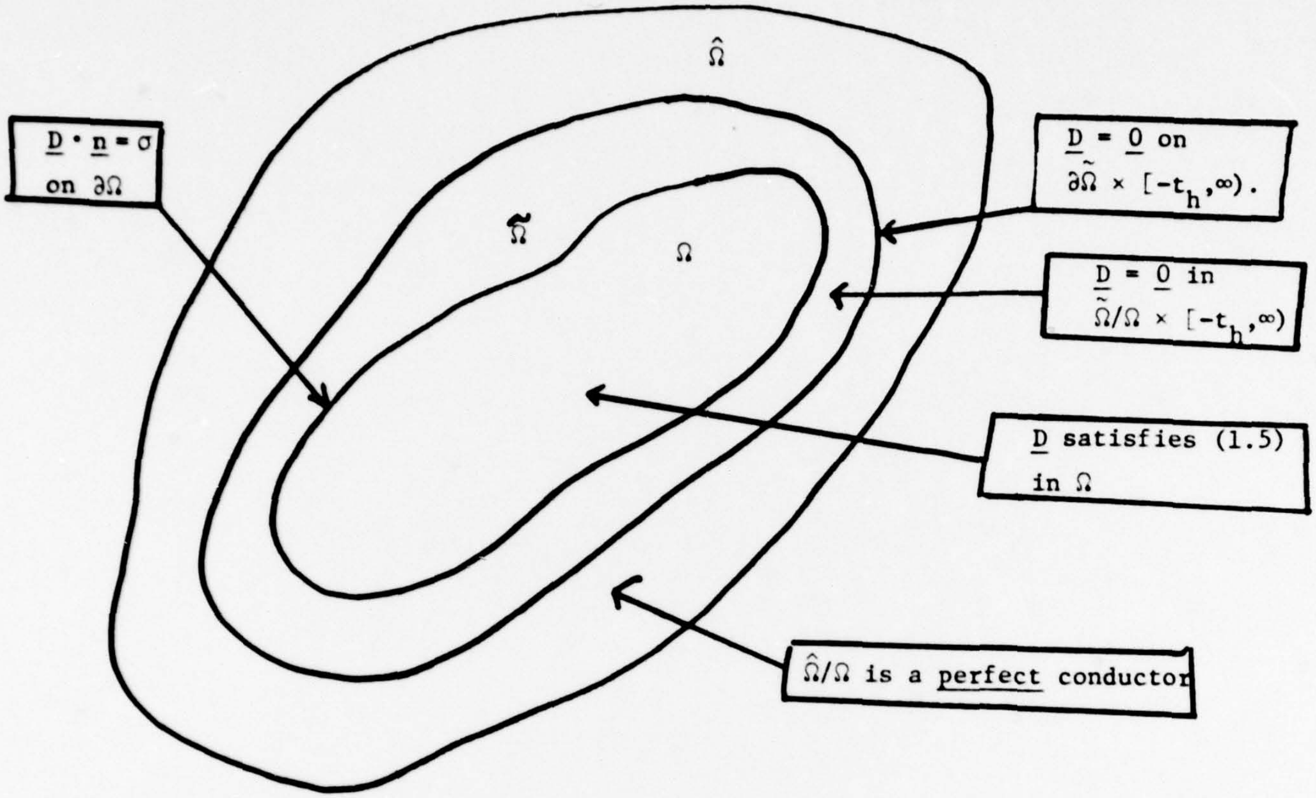
We now formulate, in a bounded domain $\tilde{\Omega} \supset \Omega$, an initial-history boundary value problem for the components of the electric displacement field: Let $\hat{\Omega} \subseteq R^3$ be a bounded domain such that $\Omega \subset \hat{\Omega}$; we assume that the region $\hat{\Omega}/\Omega$ is occupied by a perfect conductor so that $\underline{D} \equiv \underline{0}$ in $\hat{\Omega}/\Omega$ ([6],§10.5). On $\partial\Omega$, $\underline{D}(\underline{x})\cdot\underline{n}(\underline{x})=\sigma(\underline{x})$, $\underline{x} \in \partial\Omega$, where $\underline{n}(\underline{x})$ is the unit outward normal to $\partial\Omega$ at $\underline{x}$ and $\sigma(\underline{x})$ is the free charge density at $\underline{x} \in \partial\Omega$. Now let $\tilde{\Omega}$ be any bounded domain in R^3 satisfying $\Omega \subset \tilde{\Omega} \subseteq \hat{\Omega}$; then for $(\underline{x},t) \in \partial\tilde{\Omega} \times (-t_h,\infty)$, $\underline{D}(\underline{x},t)=\underline{0}$. We have, or course, equations (1.5) in $\Omega \times (0,\infty)$ and $\underline{D} = \underline{0}$ in $\tilde{\Omega}/\Omega \times (-t_h,\infty)$. In conjunction with these equations and the prescription of the past history for $(\underline{x} \in \Omega)$ given by

$$\text{(1.7a)} \qquad \underline{D}(\underline{x},t) = \begin{cases} \underline{0}, -\infty < t < -t_h \\ \underline{D}_h(\underline{x},t), \ -t_h \le t < 0, \end{cases}$$

we consider initial data of the form

$$\text{(1.7b)} \qquad \begin{cases} \underline{D}(\underline{x},0) = D_o(\underline{x}), \ \underline{x} \in \tilde{\Omega} \\ \underline{D}_t(\underline{x},0) = \underline{D}_1(\underline{x}), \ \underline{x} \in \tilde{\Omega} \end{cases}$$

where $\underline{D}_o(\underline{x})=0$ in $\tilde{\Omega}/\Omega$, $\underline{D}_1(\underline{x})=\underline{0}$ in $\tilde{\Omega}/\Omega$ and we assume that $\int_\Omega (\underline{D}_o)_i(\underline{D}_o)_i d\underline{x} \ne 0$. The situation is depicted below



2. The Initial-History Value Problem in Hilbert Space

We introduce three spaces: $H = \underline{L}_2(\tilde{\Omega})^{(1)}$ with the standard inner-product

$$\langle \underline{v},\underline{w}\rangle_{L_2} = \int_{\tilde{\Omega}} v_i w_i \, d\underline{x}$$

the Sobolev space $H_+ = \underline{H}^1_o(\tilde{\Omega})$ with inner-product

$$\langle \underline{v},\underline{w}\rangle_{\underline{H}^1_o} = \int_{\tilde{\Omega}} \frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} \, d\underline{x}$$

and $H_- = \underline{H}^{-1}(\tilde{\Omega})$, the completion of $C^\infty_o(\tilde{\Omega})$ under the norm

$$||\underline{v}||_{\underline{H}^{-1}} = \sup_{w \in \underline{H}^1_o} \left[\left|\int_{\tilde{\Omega}} v_i w_i d\underline{x}\right| / \left(\int_{\tilde{\Omega}} \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} \, d\underline{x}\right)^{\frac{1}{2}}\right]$$

It is well-known that $\underline{H}^{-1} = (\underline{H}^1_o)^{-1}$(dual space) that $\underline{H}^1_o(\tilde{\Omega}) \subseteq \underline{L}_2(\tilde{\Omega})$, both topologically

---

(1) $\underline{L}_2(\Omega) = (L_2(\tilde{\Omega}))^3$, i.e., $\underline{v} \in \underline{L}_2(\tilde{\Omega})$ iff $v_i \in L_2(\tilde{\Omega})$, $i = 1,2,3$ with similar interpretations for $\underline{H}^1_o(\tilde{\Omega})$, $\underline{H}^{-1}(\tilde{\Omega})$ introduced below.

and algebraically, and that $\underset{\sim}{H}_o^1(\tilde{\Omega})$ is dense in $\underset{\sim}{L}_2(\tilde{\Omega})$; we denote the embedding constant for the inclusion map $i : \underset{\sim}{H}_o^1(\tilde{\Omega}) \to \underset{\sim}{L}_2(\tilde{\Omega})$ by γ, so that $||\underline{v}||_{\underset{\sim}{L}_2(\tilde{\Omega})} \le \gamma ||\underline{v}||_{\underset{\sim}{H}_o^1(\tilde{\Omega})}$, $\forall \underline{v} \in \underset{\sim}{H}_o^1(\tilde{\Omega})$. Operators $\underline{N} \in L_s(\underset{\sim}{H}_o^1(\tilde{\Omega}); \underline{H}^{-1}(\tilde{\Omega}))$ and $\underline{K} \in L^2((-\infty,\infty); L_s(\underset{\sim}{H}_o^1(\tilde{\Omega}), \underline{H}^{-1}(\tilde{\Omega}))$, where $L_s(\underset{\sim}{H}_o^1(\tilde{\Omega}); H^{-1}(\tilde{\Omega}))$ denotes the space of all bounded symmetric linear operators from $\underset{\sim}{H}_o^1(\tilde{\Omega})$ into $\underline{H}^{-1}(\tilde{\Omega})$, may now be defined as follows: for any $\underline{v} \in \underset{\sim}{H}_o^1(\tilde{\Omega})$, $t \in (-\infty, \infty)$

$$(\underline{N}\underline{v})_i = \dot{\Psi}(0)[c_o \nabla^2 v_i - v_i], \quad c_o \equiv b_o/a_o \dot{\Psi}(0)$$

$$(\underline{K}(t)\underline{v})_i = \ddot{\Psi}(t) v_i - \left(\frac{b_o}{a_o}\right)\Phi(t)\nabla^2 v_i$$

where the derivatives are understood in the distribution sense, i.e., $\nabla^2 v_i = \nu_i \in \underline{L}_2(\tilde{\Omega})$ is such that for any $\phi \in C_o^\infty(\tilde{\Omega})$

$$\int_{\tilde{\Omega}} \phi\, \nu_i d\underline{x} = -\int_{\tilde{\Omega}} \frac{\partial \phi}{\partial x_j} \frac{\partial v_i}{\partial x_j} d\underline{x}.$$

The symmetry and boundedness of $\underline{N}$ and $\underline{K}(t)$, $t \in (-\infty, \infty)$, as maps of $\underset{\sim}{H}_o^1(\tilde{\Omega})$ into $\underline{H}^{-1}(\tilde{\Omega})$ will be verified in §3. If we now set $\Gamma = \Psi(0) > 0$ then with the definitions of $\underline{N}, \underline{K}(t)$ as given above the initial-boundary value problem (1.5)-(1.7) is equivalent to the following initial-history value problem in Hilbert space: find $\underline{u} \in C^2([0,\infty); \underset{\sim}{H}_o^1(\tilde{\Omega}))$ such that $\underline{u}_t \in C^1([0,\infty); \underset{\sim}{H}_o^1(\tilde{\Omega}))$, $\underline{u}_{tt} \in C([0,\infty); \underline{H}^{-1}(\tilde{\Omega}))$(2) and

$$(2.1) \qquad \begin{cases} \underline{u}_{tt} + \Gamma \underline{u}_t - \underline{N}\underline{u} + \int_{-\infty}^t \underline{K}(t-\tau)\underline{u}(\tau)d\tau = \underline{0}, \quad t > 0 \\ \underline{u}(0) = \underline{u}_o, \ \underline{u}_t(0) = \underline{v}_o \quad (\underline{u}_o, \underline{v}_o \in \underset{\sim}{H}_o^1(\tilde{\Omega})) \\ \underline{u}(\tau) = \begin{cases} \underline{0}, & -\infty < \tau < -t_h \\ \underline{U}_h(\tau), & -t_h \le \tau < 0 \end{cases} \end{cases}$$

In general, without definiteness assumptions on the operators $\underline{N}$ and $\underline{K}(t)$, $t \in (-\infty,\infty)$, this abstract initial-history value problem for $\underline{u}(t)$ is ill-posed. However, we will show that with no definiteness assumptions on $\underline{N}$ and only mild assumptions on $\underline{K}(t)$, i.e.,

(2) $\underline{u}: [0,\infty) \to \underset{\sim}{H}_o^1$ satisfying these smoothness assumptions will be called a strong solution of (2.1).

A1] $-\langle \underline{v}, \underline{K}(0)\underline{v}\rangle \geq 0, \quad \underline{v} \in \underline{H}_o^1(\tilde{\Omega})$

A2] $K(t) = ||\underline{K}(t)||_{L_s(\underline{H}_o^1;\underline{H}^{-1})}$ satisfies $K(\cdot) \in L_1[0,\infty)$

A3] $\hat{K}(t) \equiv \int ||\underline{K}_t||_{L_s(\underline{H}_o^1(\Omega);\underline{H}^{-1}(\Omega))} dt$ satisfies $\hat{K}(\cdot) \in L_1[0,\infty)$ with $\hat{K}(0) = 0$.

where $\underline{K}_t$ denotes the strong operator derivative of $\underline{K}$, it is possible to derive asymptotic lower bounds for the $\underline{L}_2$ norms of solutions $\underline{u}$ to the system (2.1) which lie in classes of bounded perturbations N of the form

(2.2) $$N = \{\underline{v} \in C([-t_h,\infty);\underline{H}_o^1) \mid \sup_{[-t_h,\infty)} ||\underline{v}||_{\underline{H}_o^1} \leq N\}$$

for some $N > 0$. Our results are obtained by using a mixture of logarithmic convexity and concavity arguments which have been used sucessfully now for over a period of more than a decade in order to treat problems of uniqueness, stability, and continuous dependence for solutions to ill-posed initial-boundary value problems and initial-history boundary value problems associated with various linear and non-linear partial differential equations and integrodifferential equations [see [7]-[9],[10]-[12], and the references cited therin]

Remarks We offer below some comments regarding previous work related to one or more aspects of the current investigation:

(i) Growth estimates for a class of damped linear integrodifferential equations associated with holohedral isotropic dielectric response have been obtained in [1] via a concavity argument; the nature of the estimates precludes our obtaining from them any information concerning the behavior of solutions as $t \to +\infty$. More specifically, we have shown the following: For any $\alpha > 0$, let $\underline{u}^{\alpha}$ be a strong solution of (2.1) with $u^{\alpha}(0) = \alpha u_o$, where it is assumed that $\langle \underline{u}_o, \underline{v}_o\rangle_{\underline{L}_2} > 0$, $\langle \underline{u}_o, \underline{N}\underline{u}_o\rangle_{\underline{L}_2} > 0$, $\langle \underline{u}_o, \int_{-t_h}^{o} \underline{K}(-\tau)\underline{U}_h(\tau)d\tau\rangle_{\underline{L}_2} < 0$, and that $\underline{K}$ satisfies hypotheses A1, A2, and $\int_0^\infty \|\underline{K}_t\|_{L_s(\underline{H}_o^1(\tilde{\Omega})\, \underline{H}^{-1}(\tilde{\Omega})} dt < \infty$. Then, provided $\|\underline{u}_o\|^2_{\underline{L}_2} \le \frac{2}{\Gamma} \langle \underline{u}_o, \underline{v}_o \rangle_{\underline{L}_2}$

and $T > \frac{1}{\Gamma} \ell n \left(\dfrac{2\langle \underline{u}_o, \underline{v}_o \rangle_{\underline{L}_2}}{2\langle \underline{u}_o, \underline{v}_o \rangle_{\underline{L}_2} - \Gamma \|\underline{u}_o\|^2_{\underline{L}_2}} \right)$ it follows that

$$(2.3) \quad \sup_{-\infty < t < T} \|\underline{u}^\alpha(t)\|_{\underline{H}_o^1(\tilde{\Omega})} \ge \frac{|\langle \underline{u}_o, \int_{t_1}^{o} \underline{K}(-\tau)\underline{U}_h(\tau)d\tau \rangle_{L_2}|^{\frac{1}{2}}}{\gamma\, \Sigma_T} \sqrt{\alpha} \quad \text{for each}$$

$\alpha \ge \|\underline{v}_o\|_{\underline{L}_2} / \langle \underline{u}_o, \underline{N}\underline{u}_o \rangle^{\frac{1}{2}}_{\underline{L}_2}$ where

$$\Sigma_T = \tfrac{1}{2}\|\underline{N}\|_{L_s(\underline{H}_o^1(\tilde{\Omega});\, \underline{H}^{-1}(\tilde{\Omega}))} + \int_0^\infty \|\underline{K}(\tau)\|_{L_s(\underline{H}_o^1(\tilde{\Omega});\, \underline{H}^{-1}(\tilde{\Omega}))} d\tau$$

$$+ T \int_0^\infty \|\underline{K}_\tau(\tau)\|_{L_s(\underline{H}_o^1(\tilde{\Omega});\, \underline{H}^{-1}(\tilde{\Omega}))} d\tau.$$

The estimate (2.3) does not require that $\underline{u}$ belong to a class of bounded pertubations of the type specified by the set N defined in (2.2) but it is limited to $T < \infty$.(3) An estimate completely anologous to (2.3) is available for the undamped situation, i.e., (2.1) with $\Gamma = 0$, but can not, in view of the hypotheses which led to (2.3) for the undamped situation, be obtained by simply setting $\Gamma = 0$ in those hypotheses. The initial-history value problem (2.1), with $\Gamma = 0$, is shown in [3] to model the evolution of the electric displacement field $\underline{D}$ in a nonconducting dielectric of Maxwell-Hopkinson type and an estimate of the type (2.3) is obtained there under the assumption that $T > \dfrac{\|\underline{u}_o\|^2_{\underline{L}_2}}{2\langle \underline{u}_o, \underline{v}_o \rangle_{\underline{L}_2}}$.

Finally, we indicate that in contrast to the various concavity arguments employed in [1] and [3], for the damped and undamped integrodifferential initial-history value problems associated with (2.1), growth estimates for solutions to these respective problems which lie in bounded classes of perturbations, of the type

(3) The estimates in [1] and [3] are obtained by using a modified concavity argument.

$\mathcal{N}$, can also be obtained by using logarithmic convexity arguments, i.e., [4]; the nature of the logarithmic convexity argument, however, involves not only a restriction to classes of bounded perturbations but also a restriction to finite time intervals of the form $[0,T)$, $T<\infty$, and requires, in addition, the stonger hypothesis that

$$-\langle \underline{v}, \underline{K}(0)\underline{v}\rangle_{\underline{L}_2} \geq \kappa ||\underline{v}||^2_{\underline{H}_o^1}, \forall \underline{v} \in \underline{H}_o^1(\tilde{\Omega})$$

with

$$\kappa \geq \gamma T \sup_{[0,T)} ||\underline{K}_t||_{L_s(\underline{H}_o^1(\tilde{\Omega});\underline{H}^{-1}(\tilde{\Omega}))}$$

Logarithmic convexity arguments have also been employed in [8] and [9], to obtain uniqueness and continuous dependence theorems, as well as growth estimates, for solutions to ill-posed initial-history boundary value problems in isothermal viscoelasticity, and in [16] to obtain growth estimates for solutions to a class of nonlinear integrodifferential equations in Hilbert space.

(ii) Several authors ([13],[14], and the references cited therein) have studied the asymptotic behavior of solutions to initial-value problems associated with damped evolution equations of the form

$$\underline{u}_{tt} + \underline{A}\underline{u}_t + \underline{B}\underline{u} = 0 \tag{2.4}$$

where $\underline{u}:[0,\infty)\to H$, a real Hilbert space with inner-product $\langle\, ,\, \rangle$ and natural norm $||(\cdot)||$; the usual assumptions which are made are that $\underline{B}$ is in $L(H;H)$ and satisfies a coerciveness condition of the form

$$\langle \underline{v}, \underline{B}\underline{v}\rangle \geq \lambda ||\underline{v}||^2, \ \lambda>0, \ \underline{v} \in \mathcal{D}(\underline{B}) \tag{2.5}$$

with $\overline{\mathcal{D}(\underline{B})} = H$. When the linear operator $\underline{A}$ satisfies $\langle \underline{A}\underline{v}, \underline{v}\rangle \geq 0$, and $\underline{A}^{-1}$ exists (the strongly damped case) it is well-known that the energy

$$E(t)=\tfrac{1}{2}(||\underline{u}_t||^2 + \langle \underline{u}(t), \underline{B}\, \underline{u}(t)\rangle)$$

decays at a uniform exponential rate; even if $\underline{A}^{-1}$ does not exist (the weakly damped case) it can be shown that in certain circumstances $\lim_{t\to\infty} E(t)=0$. In [15] we considered the system

$$(2.6)\qquad \begin{cases} \underline{u}^{\alpha}_{tt} + \Gamma u^{\alpha}_{t} - Nu^{\alpha} = 0, & \Gamma > 0,\ 0 \le t < \infty \\ \underline{u}^{\alpha}(0)=\alpha\underline{u}_o,\ \underline{u}^{\alpha}_t(0) = \underline{v}_o & (\underline{u}_o,\underline{v}_o \in \mathcal{D}(\underline{N})) \end{cases}$$

with $\alpha > 0$ and $u^{\alpha} \in C^2([0,\infty);\mathcal{D}(N))$. If $\langle \underline{v},\underline{Nv}\rangle \le -\lambda||\underline{v}||^2, \lambda > 0, \forall \underline{v} \in \mathcal{D}(\underline{N})$. (the hypothesis corresponding to (2.5)) asymptotic stability in the energy norm follows immediately; however, it is shown [15] that if $\underline{N}$ is symmetric, $\langle \underline{v},\underline{Nv}\rangle \ge 0$, $\forall \underline{v}\in D(\underline{N})$, and there exists an element $\hat{\underline{u}}_o \in \mathcal{D}(\underline{N})$ such that $\langle \hat{\underline{u}}_o,\underline{N}\hat{\underline{u}}_o\rangle > 0$, any solution of (2.6) having the requisite smoothenss must satisfy, for $\underline{u}_o = \hat{\underline{u}}_o$, and α sufficiently large

$$(2.7)\qquad \lim_{t\to\infty} ||\underline{u}^{\alpha}(t)||^2 \ge \alpha^2 ||\hat{\underline{u}}_o||^2 e^{-\Sigma_o(\alpha\ \Gamma)}$$

where $\Sigma_o(\alpha,\Gamma)$ depends on $\hat{\underline{u}}_o, v_o$ and satisfies $\lim_{\Gamma\to\infty} \Sigma_o(\alpha,\Gamma)=0$ (i.e., solutions are asymptoically bounded away from zero, for α sufficiently large, no matter how strong the damping is. The asymptotic lower bound (2.7) is obtained in [15] by employing a mixture of logarithmic concavity and convexity arguments to establish the estimate

$$(2.8)\qquad ||\underline{u}^{\alpha}(t)||^2 \ge \alpha^2 ||\hat{\underline{u}}_o||^2 \exp[\{\frac{\langle \hat{\underline{u}}_o,\underline{v}_o\rangle}{\alpha\Gamma||\hat{\underline{u}}_o||^2}\}(1-e^{-\Gamma t})]$$

for all $t \ge 0, \alpha \ge ||\underline{v}_o||/\sqrt{\langle \hat{\underline{u}}_o,\underline{N}\hat{\underline{u}}_o\rangle}$, and does not require that $\underline{u}^{\alpha}$ be a priori restricted to lie in a class of bounded perturbations; the estimate (2.8) may be easily extended to the case where $\underline{N}\in L_s(H_+.H_-)$, $\underline{u}^{\alpha}:[0,\infty)\to H_+$ where H_+,is a second Hilbert space with inner product $<,>_+$ and natural norm $||(\cdot)||_+$ such that $H_+\subseteq H$, both algebraically and topologically, and H_- is the completion of H under the norm $||(\cdot)||_-$ defined via $||\underline{w}||_- = \sup_{\underline{v}\in H_+} \frac{|<\underline{v},\underline{w},>|}{||\underline{v}||_+}$.

In particular, the system (2.1) reduces to (2.6) if $\underline{K}=\underline{0}$, $\underline{u}_o\to\alpha\underline{u}_o$, and we identify $H=\underline{L}_2(\tilde{\Omega})$ $H_+=\underline{H}^1_o(\tilde{\Omega})$, $H_-=\underline{H}^{-1}(\tilde{\Omega})$. For the system (2.1) we shall derive asymptotic lower bounds of the form (2.7) without introducing a one-parameter family of initial-data functions of the form $\alpha\underline{u}_o$, and without making any definiteness assumptions on $\underline{N}$. For definiteness hypothesis on $\underline{N},\underline{K}(t)$, which imply the existence, uniqueness, and asymptotic stability of solutions to initial-history value problems of the type (2.1) we refer the reader to [16] and [17] and the references cited therein.

3. Asymptotic Lower Bounds for Solutions

We want to show that, under an appropriate set of circumstances, solutions $\underline{u}\in N$ of the system (2.1) are asymptotically bounded away from zero, in the $\underline{L}_2$ norm, even as the damping term $\Gamma\to+\infty$. To this end we will establish the following:

Theorem Let $\underline{u}\in N$ be a strong solution of (2.1) where $\underline{N}\in L_s(\underline{H}^1_o;\underline{H}^{-1})$ and $\underline{K}\in L^2((-\infty,\infty);L_s(\underline{H}^1_o,\underline{H}^{-1}))$ such that hypothesis A1,A2, and A3 (of §1) are satisfied. If $E(0)=\frac{1}{2}||\underline{v}_o||^2_{\underline{L}_2} - <\underline{u}_o,\underline{N}\underline{u}_o>_{\underline{L}_2} < 0$ with

$$|E(0)| > \frac{3}{2}\gamma N^2[||K||_{L_1[0,\infty)} + ||\hat{K}||_{L_1[0,\infty)}] \tag{3.1}$$

then for all t, $0\le t<\infty$, and any $\beta>0$, $F(t)=||\underline{u}||^2_{\underline{L}_2}$ satisfies the differential inequality

$$FF'' - \left(\frac{\beta+1}{2\beta+1}\right)F'^2 \ge -\Gamma FF' \tag{3.2}$$

<u>Proof</u>. From the definition of $F(t)$ we have $F' = 2<\underline{u},\underline{u}_t>_{\underline{L}_2}$ and $F'' = 2||\underline{u}_t||^2_{\underline{L}_2} + 2<\underline{u},\underline{u}_{tt}>_{\underline{L}_2}$. Direct computation then yields

$$FF'' - (\beta+1)F'^2 = 4(\beta+1)S^2_\beta + 2F\{<\underline{u},\underline{u}_{tt}>_{\underline{L}_2} - (2\beta+1)||\underline{u}_t||^2_{\underline{L}_2}\} \tag{3.3}$$

where

$$S^2_\beta(t) = ||\underline{u}||^2_{\underline{L}_2}\,||\underline{u}_t||^2_{\underline{L}_2} - <\underline{u},\underline{u}_t>_{\underline{L}_2} \geq 0 \tag{3.4}$$

by the Schwarz inequality. Therefore, for $0 \leq t < \infty$, and any $\beta > 0$

$$FF'' - (\beta+1)F'^2 \geq 2FG_\beta \tag{3.5}$$

where, in view of the integrodifferential equation (2.1_1) for $\underline{u}(t)$

$$\begin{aligned} G_\beta(t) = &<\underline{u},\underline{Nu}>_{\underline{L}_2} - \Gamma<\underline{u},\underline{u}_t>_{\underline{L}_2} - (2\beta+1)||\underline{u}_t||^2_{\underline{L}_2} \\ &- <\underline{u}, \textstyle\int^t_{-\infty}\underline{K}(t-\tau)\underline{u}(\tau)d\tau>_{\underline{L}_2} \end{aligned} \tag{3.6}$$

As $F'(t) = 2<\underline{u},\underline{u}_t>_{\underline{L}_2}$ we may rewrite (3.6) as

$$\begin{aligned} G_\beta(t) &= -\frac{\Gamma}{2}F' - (2\beta+1)[||\underline{u}_t||^2_{\underline{L}_2} - <\underline{u},\underline{Nu}>_{\underline{L}_2}] \\ &\quad -2\beta<\underline{u},\underline{Nu}>_{\underline{L}_2} - <\underline{u},\textstyle\int^t_{-\infty}\underline{K}(t-\tau)\underline{u}(\tau)d\tau>_{\underline{L}_2} \\ &= -\frac{\Gamma}{2}F' - 2(2\beta+1)E(t) - 2\beta<\underline{u},\underline{Nu}>_{\underline{L}_2} \\ &\quad - <\underline{u},\textstyle\int^t_{-\infty}\underline{K}(t-\tau)\underline{u}(\tau)d\tau>_{\underline{L}_2} \end{aligned} \tag{3.7}$$

in view of the definition of $E(t)$. Taking the $\underline{L}_2$ inner-product of (2.1_1) with $\underline{u}_t$ and integrating we easily obtain

$$\begin{aligned} E(t) = &E(0) - \Gamma\textstyle\int^t_o||\underline{u}_\tau||^2_{\underline{L}_2}d\tau \\ &- \textstyle\int^t_o<\underline{u}_\tau,\int^\tau_{-\infty}\underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda>_{\underline{L}_2}d\tau \end{aligned} \tag{3.8}$$

and substitution into (3.7_2) then yields

(3.9)
$$G_\beta(t) \geq -\frac{\Gamma}{2}F' - 2(2\beta+1)E(0) - 2\beta\langle \underline{u}, \underline{Nu}\rangle_{\underline{L}_2}$$
$$+ 2(2\beta+1)\int_0^t \langle \underline{u}_\tau, \int_{-\infty}^\tau \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda\rangle_{\underline{L}_2} d\tau$$
$$- \langle \underline{u}, \int_{-\infty}^t \underline{K}(t-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2},$$

where we have dropped a non-negative term proportional to $\int_0^t ||\underline{u}_\tau||^2_{\underline{L}_2} d\tau$. If we now take the $\underline{L}_2$ inner-product of (2.1_1) with $\underline{u}(t)$ and use the definition of $F(t)$ we obtain the identity

(3.10)
$$\tfrac{1}{2}F'' + \frac{\Gamma}{2}F' = ||\underline{u}_t||^2_{\underline{L}_2} + \langle \underline{u}, \underline{Nu}\rangle_{\underline{L}_2}$$
$$- \langle \underline{u}, \int_{-\infty}^t \underline{K}(t-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}.$$

which implies that

(3.11)
$$-2\beta\langle \underline{u}, \underline{Nu}\rangle_{\underline{L}_2} = -\beta F'' - \beta\Gamma F' + 2\beta||\underline{u}_t||^2_{\underline{L}_2}$$
$$-2\beta\langle \underline{u}, \int_{-\infty}^t \underline{K}(t-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}$$

Substituting from (3.11) into (3.9), collecting terms, and dropping a non-negative expression proportional to $||\underline{u}_t||^2_{\underline{L}_2}$ now yields the following estimate for $G_\beta(t)$:

(3.12)
$$G_\beta(t) \geq -\Gamma(\beta+\tfrac{1}{2})F' - \beta F'' - 2(2\beta+1)E(0)$$
$$-(2\beta+1)\langle \underline{u}, \int_{-\infty}^t \underline{K}(t-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}$$
$$+2(2\beta+1)\int_0^t \langle \underline{u}_\tau, \int_{-\infty}^t \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda\rangle_{\underline{L}_2} d\tau$$

Substitution for $G_\beta(t)$ from (3.12) into the differential inequality (3.5) now produces

(3.13)
$$FF'' - (\beta+1)F'^2 \geq -2\Gamma(\beta+\tfrac{1}{2})FF' - 2\beta FF''$$
$$-4(2\beta+1)E(0)F$$
$$-2(2\beta+1)F<\underline{u}, \int_{-\infty}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau>_{\underline{L}_2}$$
$$+4(2\beta+1)F\int_0^t <\underline{u}_\tau, \int_{-\infty}^{\tau} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda>_{\underline{L}_2} d\tau$$

which is equivalent to

(3.14)
$$FF'' - \left(\frac{\beta+1}{2\beta+1}\right)F'^2 \geq -\Gamma FF'$$
$$-4FE(0) - 2F<\underline{u}, \int_{-\infty}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau>_{\underline{L}_2}$$
$$+4F\int_0^t <\underline{u}_\tau, \int_{-\infty}^{\tau} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda>_{\underline{L}_2} d\tau$$

or, in view of our hypotheses that $E(0) < 0$

(3.15)
$$FF'' - \left(\frac{\beta+1}{2\beta+1}\right)F'^2 \geq -\Gamma FF'$$
$$+ 2F[2|E(0)| - <\underline{u}, \int_{-\infty}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau>_{\underline{L}_2}$$
$$+ 2\int_0^t <\underline{u}_\tau, \int_{-\infty}^{t} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\tau>_{\underline{L}_2} d\tau$$

We now seek to bound the two expressions involving $\underline{K}(t)$ on the right-hand side of (3.15). Let us first note, however, that as

(3.16)
$$<\underline{u}_\tau, \int_{-\infty}^{\tau} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\tau>_{\underline{L}_2} =$$
$$\frac{d}{d\tau}<\underline{u}(\tau), \int_{-\infty}^{\tau} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda>_{\underline{L}_2}$$
$$-<\underline{u}(\tau), \int_{-\infty}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda>_{\underline{L}_2}$$
$$-\ \underline{u}(\tau), \underline{K}(0)\underline{u}(\tau)>_{\underline{L}_2}$$

(3.15) has the equivalent form

$$(3.17)\qquad FF''-\left(\frac{\beta+1}{2\beta+1}\right)F'^2 \ge -\Gamma FF'$$
$$+\ 2F[2|E(0)|-2\int_0^t \langle \underline{u}(\tau),\underline{K}(0)\underline{u}(\tau)\rangle_{\underline{L}_2}\,d\tau$$
$$-\ 2\int_0^t \langle \underline{u}(\tau),\int_{-\infty}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda\rangle_{\underline{L}_2}\,d\tau$$
$$-\ 2\langle \underline{u}_o,\int_{-\infty}^{o} \underline{K}(-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}$$
$$+\langle \underline{u},\int_{-\infty}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}]$$

from which it follows that

$$(3.18)\qquad FF''-\left(\frac{\beta+1}{2\beta+1}\right)F'^2 \ge -\Gamma FF'$$
$$+\ 2F[2|E(0)|-2\langle \underline{u}_o,\int_{-\infty}^{o} \underline{K}(-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}$$
$$-\ 2\int_0^t \langle \underline{u}(\tau),\int_{-\infty}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda\rangle_{\underline{L}_2}\,d\tau$$
$$+\langle \underline{u},\int_{-\infty}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}$$

by virtue of hypothesis A1 relative to $\underline{K}(0)$. We now have the following estimates for the integrals appearing on the right-hand side of (3.18):

$$(i)\qquad |\langle \underline{u}_o,\int_{-\infty}^{o} \underline{K}(-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}|$$
$$\le ||\underline{u}_o||_{\underline{L}_2}\int_{-\infty}^{o} ||\underline{K}(-\tau)||_{L_s(\underline{H}_o^1;\underline{H}^{-1})}||\underline{u}(\tau)||_{\underline{H}_o^1}\,d\tau$$
$$\le \gamma ||\underline{u}_o||_{\underline{H}_o^1[-t_h,0)}\sup ||\underline{u}||_{\underline{H}_o^1}\int_{-\infty}^{o} ||\underline{K}(-\tau)||_{L_s(\underline{H}_o^1,\underline{H}^{-1})}\,d\tau$$
$$\le \gamma \Big(\sup_{[-t_h,\infty)} ||\underline{u}(t)||_{\underline{H}_o^1}\Big)^2\int_{o}^{\infty} ||\underline{K}(\tau)||_{L_s(\underline{H}_o^1,\underline{H}^{-1})}\,d\tau$$
$$\le \gamma\ N^2\int_{o}^{\infty} ||\underline{K}(\tau)||_{L_s(\underline{H}_o^1,\underline{H}^{-1})}\,d\tau$$

therefore,

(3.19)
$$-2\langle \underline{u}_o, \int_{-\infty}^{o} \underline{K}(-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2} \geq -2\gamma N^2 \int_o^\infty ||\underline{K}(t)||_{\mathcal{L}_s(\underline{H}_o^1, \underline{H}^{-1})} d\tau$$

(ii)
$$|\langle \underline{u}, \int_{-\infty}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2}|$$
$$\leq ||\underline{u}(t)||_{\underline{L}_2} \int_{-\infty}^{t} ||\underline{K}(t-\tau)||_{\mathcal{L}_s(\underline{H}_o^1, \underline{H}^{-1})} ||\underline{u}(\tau)||_{\underline{H}_o^1} d\tau$$
$$\leq \gamma \Big(\sup_{[-t_h,\infty)} ||\underline{u}(t)||_{\underline{H}_o^1}\Big)^2 \int_{-\infty}^{t} ||\underline{K}(t-\tau)||_{\mathcal{L}_s(\underline{H}_o^1, \underline{H}^{-1})} d\tau$$
$$= \gamma \Big(\sup_{[-t_h,\infty)} ||\underline{u}(t)||_{\underline{H}_o^1}\Big)^2 \int_o^\infty ||\underline{K}(\rho)||_{\mathcal{L}_s(\underline{H}_o^1, \underline{H}^{-1})} d\rho$$
$$\leq \gamma N^2 \int_o^\infty ||\underline{K}(t)||_{\mathcal{L}_s(\underline{H}_o^1, \underline{H}^{-1})} dt$$

and, therefore, for $0 \leq t < \infty$,

(3.20)
$$\langle \underline{u}, \int_{-\infty} \underline{K}(t-\tau)\underline{u}(\tau)d\tau\rangle_{\underline{L}_2} \geq -\gamma N^2 \int_o^\infty ||\underline{K}(t)||_{\mathcal{L}_s(\underline{H}_o^1, H^{-1})} d\tau$$

Finally, we have

(iii)
$$|\int_o^t \langle \underline{u}(\tau), \int_{-\infty}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda\rangle_{\underline{L}_2} d\tau|$$
$$\leq \int_o^t |\langle \underline{u}(\tau), \int_{-t_h}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda\rangle_{\underline{L}_2}| d\tau$$
$$\leq \int_o^\infty (||\underline{u}(\tau)||_{\underline{L}_2} \int_{-t_h}^{\tau} ||\underline{K}_\tau(\tau-\lambda)||_{\mathcal{L}_s(\underline{H}_o^1, \underline{H}^{-1})} ||\underline{u}(\lambda)||_{\underline{H}_o^1} d\lambda) d\tau$$
$$\leq \gamma \Big(\sup_{[-t_h,\infty)} ||\underline{u}(t)||_{\underline{H}_o^1}\Big)^2 \int_o^\infty \int_{-t_h}^{\tau} ||\underline{K}_\tau(\tau-\lambda)||_{\mathcal{L}_s(\underline{H}_o^1, \underline{H}^{-1})} d\lambda\, d\tau$$
$$= \gamma \Big(\sup_{[-t_h,\infty)} ||\underline{u}(t)||_{\underline{H}_o^1}\Big)^2 \int_o^\infty \int_o^{\tau+t_h} ||\underline{K}_\rho(\rho)||_{\mathcal{L}_s(\underline{H}_o^1, \underline{H}^{-1})} d\rho\, d\tau$$
$$\leq \gamma N^2 \int_o^\infty (\hat{k}(\tau)|_o^{\tau+t_h}) d\tau$$
$$= \gamma N^2 \int_o^\infty \hat{k}(\tau+t_h) d\tau$$
$$= \gamma N^2 \int_{t_h}^\infty \hat{k}(\lambda)d\lambda \leq \gamma N^2 ||\hat{k}||_{L_1[0,\infty)}.$$

where $\hat{K}(\lambda)=\int ||\underline{K}_\lambda(\lambda)||_{L_s(\underline{H}_o^1,\underline{H}^{-1})}d\lambda$. Therefore, for $0 \le t < \infty$,

$$-2\int_o^t <\underline{u}(\tau),\int_{-\infty}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda>_{\underline{L}_2} d\tau \tag{3.21}$$

$$\ge -2\gamma\cdot N^2 ||\hat{K}||_{L_1[0,\infty)}.$$

Combining (3.18) with (3.19)-(3.21) then yields the estimate

$$FF''-\left(\frac{\beta+1}{2\beta+1}\right)F'^2 \ge -\Gamma FF' \tag{3.22}$$

$$+ 2F[2|E(0)| - 3\gamma N^2\{||K||_{L_1[0,\infty)}+||\hat{K}||_{L_1[0,\infty)}\}]$$

which, in view of our hypothesis relative to $|E(0)|$, implies the stated inequality, i.e. (3.2).

<u>Corollary 1</u>. Under the same conditions which prevail in the Theorem above

$$\lim_{t\to\infty} ||\underline{u}(t)||^2_{\underline{L}_2} > ||\underline{u}_o||^2_{\underline{L}_2} \exp\left(\frac{2<\underline{u}_o,\underline{v}_o>_{\underline{L}_2}}{\Gamma||\underline{u}_o||^2_{\underline{L}_2}}\right) \tag{3.23}$$

<u>Proof</u>: In (3.2), which is valid for all $\beta > 0$, we take the limit $\beta \to 0^+$ and obtain

$$FF''-F'^2 \ge -\Gamma FF', \quad 0 \le t < \infty. \tag{3.24}$$

Direct integration of this differential inequality then yields the lower bound

$$F(t) \ge F(0)\exp\left[\left(\frac{F'(0)}{\Gamma F(0)}\right)(1-e^{-\Gamma t})\right], \quad 0 \le t < \infty \tag{3.25}$$

which, in turn, implies that

$$\lim_{t\to+\infty} F(t) \ge F(0)\exp\left(\frac{F'(0)}{\Gamma F(0)}\right) \tag{3.26}$$

This last result is equivalent, via the definition of F(t), to (3.23).

<u>Q.E.D.</u>

A better lower bound and asymptotic estimate (as $t\to+\infty$) may be obtained with a little further effort, namely, we have

<u>Corollary 2</u>. Under the same conditions which prevailed in the above Theorem, it follows that for all $t > 0$, and any α, $\frac{1}{2} < \alpha < 1$,

(3.27a) $$||\underline{u}||^2_{\underline{L}_2} \geq ||\underline{u}_o||^2_{\underline{L}_2}\left[1+\left(\frac{2(1-\alpha)<\underline{u}_o,\underline{v}_o>_{\underline{L}_2}}{\Gamma||\underline{u}_o||^2_{\underline{L}_2}}\right)(1-e^{-\Gamma t}\right]^{\frac{1}{1-\alpha}}$$

so that, as $t\to t+\infty$

(3.27b) $$\lim_{t\to+\infty} ||\underline{u}(t)||^2_{\underline{L}_2} \geq ||\underline{u}_o||^2_{\underline{L}_2}\left[1+\frac{2(1-\alpha)<\underline{u}_o,\underline{v}_o>_{\underline{L}_2}}{\Gamma||\underline{u}_o||^2_{\underline{L}_2}}\right]^{\frac{1}{1-\alpha}}$$

<u>Proof</u>. For any $\alpha > 0$

(3.28) $$[F^{(1-\alpha)}]''(t) = (1-\alpha)F^{-\alpha-1}(t)[F(t)F''(t)-\alpha F'^2(t)$$

and from (3.2) with $\alpha = \frac{\beta+1}{2\beta+1}$

(3.29) $$(1-\alpha)F^{-\alpha-1}[FF''-\alpha F'^2] \geq (1-\alpha)F^{-\alpha-1}[-\Gamma FF']$$
$$= -\Gamma(1-\alpha)F^{-\alpha}F'$$

Therefore for $\alpha = \frac{\beta+1}{2\beta+1}$

(3.30) $$[F^{(1-\alpha)}]''(t) \geq -\Gamma(1-\alpha)F^{-\alpha}F'$$
$$= -\Gamma[F^{(1-\alpha)}]'(t)$$

Let $G(t)=F^{(1-\alpha)}(t)$ and $H(t)=G'(t)$; then (3.30) implies that $H'(t)\geq-\Gamma H(t)$ and an integration produces

$$H(t)\geq H(0)e^{-\Gamma t} \leftrightarrow G'(t)\geq G'(0)e^{-\Gamma t}$$

A second integration then yields

$$G(t) \geq G(0) + \frac{G'(0)}{\Gamma}[1-e^{-\Gamma t}]$$

which is equivalent to

(3.31) $$F^{(1-\alpha)}(t) \geq F^{(1-\alpha)}(0) + \frac{(1-\alpha)F^{-\alpha}(0)F'(0)}{\Gamma}(1-e^{-\Gamma t})$$

$$= F^{(1-\alpha)}(0)[1+\frac{(1-\alpha)F'(0)}{\Gamma F(0)}(1-e^{-\Gamma t})]$$

from which the stated estimate (3.27a) follows after taking the $(1-\alpha)$th root on both sides of (3.31) and using the definition of $F(t)$; we note that (3.27b) follows directly from this last estimate and that $\alpha=\beta+1/2\beta+1$ takes on all values in the in the interval $(\frac{1}{2},1)$ for $\beta > 0$.

Q.E.D.

Remark Clearly as $\beta\to 0^+, \alpha\to 1$; taking the limit in (3.27b) as $\alpha\to 1$ and using the elementary fact that

$$\lim_{\lambda\to 0}[1+\lambda x]^{\frac{1}{\lambda}} = e^x$$

we recover (3.23) from (3.27b).

Remark Clearly both (3.23) and (3.27b) imply that

$$\lim_{\Gamma\to+\infty}\lim_{t\to+\infty}||\underline{u}(t)||^2_{\underline{L}_2} \geq ||\underline{u}_o||^2_{\underline{L}_2}$$

so that the $\underline{L}_2$ norm of $\underline{u}$ is bounded from below as $t\to+\infty$ even as the damping becomes arbitrarily large; this is the analogue, for the ill-posed integrodifferential initial-history value problem (2.1), of the asymptotic lower bound obtained in [15].

Remark We comment here on some of the conditions imposed by the hypothesis of the Theorem on the electromagnetic memory functions Φ and Ψ which appear in (1.5) and serve, therefore, to define the operators $\underline{N}$ and $\underline{K}(t)$ we have[(4)]

(3.32)
$$\begin{aligned}
\langle \underline{v},\underline{K}(0)\underline{v}\rangle_{\underline{L}_2} &= \int_{\tilde\Omega} v_i[\underline{K}(0)\underline{v}]_i\, d\underline{x}\\
&= \ddot\Psi(0)\int_{\tilde\Omega} v_i v_i d\underline{x} - \left(\frac{b_o}{a_o}\right)\Phi(0)\int_{\tilde\Omega} v_i\nabla^2 v_i\, d\underline{x}\\
&\ddot\Psi(0)\,||\underline{v}||^2_{\underline{L}_2} - \left(\frac{b_o}{a_o}\right)\Phi(0)\Big[\int_{\partial\tilde\Omega} v_i\frac{\partial v}{\partial x_j} n_j\, d\underline{x} - \\
&\qquad \int_{\tilde\Omega}\frac{\partial v_i}{\partial x_i}\frac{\partial v_i}{\partial x_j}\, d\underline{x}\Big]\\
&= \ddot\Psi(0)\,||\underline{v}||^2_{\underline{L}_2} + \left(\frac{b_o}{a_o}\right)\Phi(0)\,||\underline{v}||^2_{\underline{H}^1_o}
\end{aligned}$$

for any $\underline{v}\in\underline{H}^1_o(\tilde\Omega)$. Therefore (hypothesis A1) $-\langle\underline{v},\underline{K}(0)\underline{v}\rangle_{\underline{L}_2}\geq 0$, $\forall\,\underline{v}\in\underline{H}^1_o(\tilde\Omega)$, iff

(3.33)
$$\ddot\Psi(0)\,||\underline{v}||^2_{\underline{L}_2} + \left(\frac{b_o}{a_o}\right)\Phi(0)\,||\underline{v}||^2_{\underline{H}^1_o} \leq 0$$

If $\ddot\Psi(0)\geq 0$ then via the embedding of $\underline{H}^1_o(\tilde\Omega)$, into $\underline{L}_2(\tilde\Omega)$

$$\ddot\Psi(0)\,||\underline{v}||^2_{\underline{L}_2} \leq \gamma^2\ddot\Psi(0)\,||\underline{v}||^2_{\underline{H}^1_o}$$

and (3.33) will be satisfied, for all $\underline{v}\in\underline{H}^1_o(\tilde\Omega)$, provided

$$\gamma^2\ddot\Psi(0) + \left(\frac{b_o}{a_o}\right)\Phi(0)\leq 0 \;\leftrightarrow\; \Phi(0)\leq -\left(\frac{a_o}{b_o}\right)\gamma^2\ddot\Psi(0)$$

Thus, as far as hypothesis A1 goes, we have

(3.34)
$$\left\{\ddot\Psi(0)\geq 0,\ \Phi(0)\leq -\left(\frac{a_o}{b_o}\right)\gamma^2\ddot\Psi(0)\right\} \Longrightarrow$$
$$-\langle\underline{v},\underline{K}(0)\underline{v}\rangle_{\underline{L}_2}\geq 0,\ \forall\,\underline{v}\in\underline{H}^1_o(\tilde\Omega)$$

In view of (3.33), the same conclusion obtains if $\ddot\Psi(0)\leq 0$, $\Phi(0)\leq 0$. Also

(3.35)
$$\begin{aligned}
||\underline{K}(t)||_{L_s(\underline{H}^2_o,\underline{H}^{-1})} &= \sup_{\underline{v}\in\underline{H}^1_o}\frac{|\langle\underline{v},\underline{K}(t)\underline{v}\rangle_{\underline{L}_2}|}{||\underline{v}||_{\underline{H}^1_o}}\\
&= \sup_{\underline{v}\in\underline{H}^1_o}\frac{|\int_{\tilde\Omega} v_i[\underline{K}(t)v_i]d\underline{x}|}{||\underline{v}||_{\underline{H}^1_o}}
\end{aligned}$$

[(4)] To be consistent with the formulation of the initial-history boundary value problem in §1 we have, in fact, $\underline{v}=\underline{0}$ in $\tilde\Omega/\Omega$ in the computation below.

$$= \sup_{\underline{v}\in \underline{H}_o^1} \frac{|\ddot{\Psi}(t)|\,||\underline{v}||^2_{\underline{L}_2} + (\frac{b_o}{a_o})\Phi(t)\,||\underline{v}||^2_{\underline{H}_o^1}|}{||\underline{v}||^2_{H_o^1}}$$

$$\leq \sup_{\underline{v}\in \underline{H}_o^1} \left(\frac{|\ddot{\Psi}(t)|\,||\underline{v}||^1_{\underline{L}_2}}{||\underline{v}||^2_{\underline{H}_o^1}} \right) + (\frac{b_o}{a_o})\Phi(t)$$

$$\leq \gamma^2|\ddot{\Psi}(t)| + (\frac{b_o}{a_o})|\Phi(t)|$$

Clearly, hypothesis A2 will then be satisfied if $\int_o^\infty |\ddot{\Psi}|dt < \infty$, and $\int_o^\infty |\Phi(t)|dt < \infty$, i.e.

(3.36) $$\{|\ddot{\Psi}|\in L_1[0,\infty), \ |\Phi|\in L_1[0,\infty)\} \rightarrow K \in L_1[0,\infty).$$

A computation entirely analogous to (3.35) yields.

(3.37) $$||\underline{K}_t||_{L_s(\underline{H}_o^1, \underline{H}^{-1})} \leq \gamma^2|\Psi^{(3)}(t)| + (\frac{b_o}{a_o})|\dot{\Phi}(t)|$$

and, therefore, for hypothesis A3 we have

(3.38) $$\int|\Psi^{(3)}|dt\in L_1[0,\infty), \quad \int|\dot{\Phi}|dt\in L_1[0,\infty)$$

$$\int|\Psi^{(3)}|dt|_{t=0}, =0 \ \int|\dot{\Phi}|(t)|dt|_{t=0}=0$$

$$\hat{K}\in \underline{L}_1[0,\infty) \text{ with } \hat{K}(0)=0$$

Finally, for any $\underline{v}\in\underline{H}_o^1(\tilde{\Omega})$, we have

(3.39) $$\langle \underline{v},\underline{Nv}\rangle_{\underline{L}_2} = \int_{\tilde{\Omega}} v_i[\underline{Nv}]_i d\underline{x}$$

$$= \dot{\Psi}(0)[\int_{\tilde{\Omega}} c_o v_i \nabla^2 v_i d\underline{x} - \int_{\tilde{\Omega}} v_i v_i d\underline{x}]$$

$$= c_o\dot{\Psi}(0)[\int_{\partial\tilde{\Omega}} v_i \frac{\partial v_i}{\partial x_j} n_j d\underline{x} - \int_{\tilde{\Omega}} \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\underline{x}]$$

$$- \dot{\Psi}(0)||\underline{v}||^2_{\underline{L}_2}$$

$$= -\dot{\Psi}(0)[c_o||\underline{v}||^2_{\underline{H}_o^1} + ||\underline{v}||^2_{\underline{L}_2}]$$

therefore,

$$(3.40)\qquad \begin{aligned} 2E(0) &= ||\underset{\sim}{v}_o||^2_{\underline{L}_2} - \langle \underset{\sim}{u}_o, N\underset{\sim}{u}_o \rangle_{\underline{L}_2} \\ &= ||\underset{\sim}{v}_o||^2_{\underline{L}_2} + \dot{\Psi}(0)[c_o||\underset{\sim}{u}_o||^2_{\underset{\sim}{H}^1_o} + ||\underset{\sim}{u}_o||^2_{\underline{L}_2}] \\ &= ||\underset{\sim}{v}_o||^2_{\underline{L}_2} + (\frac{b_o}{a_o})||\underset{\sim}{u}_o||^2_{\underset{\sim}{H}^1_o} + \dot{\Psi}(0)||\underset{\sim}{u}_o||^2_{\underline{L}_2} < 0 \end{aligned}$$

iff

$$(3.41)\qquad \dot{\Psi}(0) < -[||\underset{\sim}{v}_o||^2_{\underline{L}_2} + (\frac{b_o}{a_o})||\underset{\sim}{u}_o||^2_{\underset{\sim}{H}^1_o}] \Big/ ||\underset{\sim}{u}_o||^2_{\underline{L}_2}$$

If $\dot{\Psi}(0)$ satisfies (3.41) then

$$(3.41)\qquad |E(0)| = \tfrac{1}{2}[|\dot{\Psi}(0)|\,||\underset{\sim}{u}_o||^2_{\underline{L}_2} - (||\underset{\sim}{v}_o||^2_{\underline{L}_2} + \left(\frac{b_o}{a_o}\right)||\underset{\sim}{u}_o||^2_{H^1_o})]$$

and (3.1) is equivalent to requiring that

$$(3.42)\qquad \begin{aligned} |\dot{\Psi}(0)| \geq \frac{1}{||\underset{\sim}{u}_o||^2_{\underline{L}_2}} \Bigg(& 3\gamma N^2(||K||_{L_1[0,\infty)} + ||\hat{K}||_{L_1[0,\infty)}) \\ & + ||\underset{\sim}{v}_o||^2_{\underline{L}_2} + (\frac{b_o}{a_o})||\underset{\sim}{u}_o||^2_{\underset{\sim}{H}^1_o} \Bigg) \end{aligned}$$

where

$$(3.43)\qquad \begin{aligned} ||K||_{L_1[0,\infty)} &\leq \gamma^2\int_o^\infty|\ddot{\Psi}(t)|dt + \left(\frac{b_o}{a_o}\right)\int_o^\infty|\Phi(t)|dt \\ ||\hat{K}||_{L_1[0,\infty)} &\leq \gamma^2\int_o^\infty\int_o^\tau|\Psi^{(3)}(\lambda)|d\lambda d\tau \\ &\quad + \left(\frac{b_o}{a_o}\right)\int_o^\infty\int_o^\tau|\dot{\Phi}|d\lambda d\tau \end{aligned}$$

by (3.35), (3.37), and the definitions of $K(\cdot)$ and $\hat{K}(\cdot)$.

REFERENCES

1. Bloom, F., "Concavity Arguments and Growth Estimates for Damped Linear Integrodifferential Equations with Applications to a Class of Holohedral Isotropic Dielectrics", ZAMP, vol. 29, (1978), 644-663.

2. Toupin, R. A. and R. S. Rivlin, "Linear Functional Electromagnetic Constitutive Relations and Plane Waves in a Hemihedral Isotropic Material", Arch. Rat. Mech. Anal., Vol. 6, (1960), 188-197.

3. Bloom, F. "Growth Estimates for Electric Displacement Fields and Bounds for Constitutive Constants in the Maxwell-Hopkinson Theory of Dielectrics", Int. J. Eng. Sci. (in press).

4. Bloom F., "Stability and Growth Estimates for Electric Fields in Nonconducting Material Dielectrics", J. Math. Anal. Applic., vol. 67, (1979).

5. Volterra, V., Theory of Functionals, Dover Press, N. Y.

6. Cook, D. M., The Theory of the Electromagnetic Field, Prentice-Hall, Inc., N. J.

7. Bloom, F., "Stability and Growth Estimates for Volterra Integrodifferential Equations in Hilbert Space", Bull. A.M.S., vol. 82, #4, July 1976.

8. Bloom, F., "Growth Estimates for Solutions ot Initial-Boundary Value Problems in Viscoelasticity", J. Math. Anal. Applic., vol. 59, (1977), 469-478.

9. Bloom, F., "Continuous Data Dependence for an Abstract Volterra Integrodifferential Equations in Hilbert Space with Applications to Viscoelasticity", Annali della Scuola Normale (PISA), vol. IV, #1, (1977), 179-207.

10. Levine, H. A. and L. E. Payne, "Nonexistence Theorems for the Heat Equation with Nonlinear Boundary Conditions and for the Porous Medium Equation Backward in Time", J. Diff. Eqs., vol. 16, (1974), 319-334.

11. Levine, H. A., "Instability and Nonexistence of Global Solutions to Nonlinear Wave Equations of the Form $Pu_{tt} = -Au + F(u)$", Trans. A.M.S., vol. 192, (1974), 1-21.

12. Levine, H. A. and L. E. Payne, "On the Nonexistence of Entire Solutions to Nonlinear Second-Order Elliptic Equations", SIAM. J. Math. Anal., vol. 7 (1976), 337-342.

13. Russell, D. L., "Decay Rates for Weakly Damped Systems in Hilbert Space Obtained with Control-Theoretic Methods", J. Diff. Eqs. vol. 19, (1975), 344-370.

14. Dafermos, C. M., "Wave Equations with Weak Damping", SIAM. J. Appl. Math, vol. 18, (1970), 759-767.

15. Bloom, F., "Remarks on the Asymptotic Behavior of Solutions to Damped Evolution Equations in Hilbert Space", Proc. A.M.S. (in press).

16. Dafermos, C. M., "Contraction Semigroups and the Trend to Equilibrium in Continuum Mechanics," Proceedings I.U.T.A.M./I.M.U. Conference on Applications of Functional Analysis to Mechanics (1975).

17. Dafermos, C. M., "An Abstract Volterra Equations with Applications to Linear Viscoelasticity", J. Diff. Eqs. vol. 7 (1970), 544-569.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER (19) AFOSR-TR- 79-0723 (18)	2. GOVT ACCESSION NO.
	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) Asymptotic Bounds for Solutions to a System of Damped Integrodifferential Equations of Electromagnetic Theory	5. TYPE OF REPORT & PERIOD COVERED (9) Interim repti
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Frederick Bloom	8. CONTRACT OR GRANT NUMBER(s) (15) AFOSR-77-3396
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of South Carolina Math, Computer Science, and Statistics Columbia, SC 29208	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F (16) 2304/A4 (17) A4
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research NM Bolling AFB, Washington DC 20332	12. REPORT DATE 5/28/79
	13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 27 p.	15. SECURITY CLASS. (of this report) Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

(11) 28 May 79

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Accepted by: J. Math. Analysis and Applications

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Holohedral Isotropic Dielectric; damped system of integrodifferential equations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For the system of damped integrodifferential equations which govern the evolution of the electric induction field in a class of rigid holohedral isotropic dielectrics of the type

410 443

Unclassified

introduced by Toupin and Rivlin, conditions on the memory functions are deduced which imply that the L_2 norms of such induction fields are bounded away from zero even as the damping grows in an unbounded manner; explicit lower bounds f the L_2 norms of the induction fields in such dielectrics are derived as $t \to +\infty$.

increases without limit.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)